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MINIMIZING EXPECTED SHORTAGES

IN A MULTI-ITEM INVENTORY SYSTEM

by

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ABSTRACT:

A formulation is given for minimizing the expected total number of shortages in a multi-item inventory system under a continuous review policy. The variable reorder levels and order quantities must be set to meet constraints on the average expected investment in on hand inventory and on the expected number of orders placed per unit time. Several approaches to the problem are presented.

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MINIMIZING EXPECTED SHORTAGES IN A

MULTI-ITEM INVENTORY SYSTEM

I. Introduction and Problem Statement.

This report deals with a multi-item inventory problem arising in the management of the Navy's wholesale inventory control system. The problem involves an extremely large number of items, as many as one hundred thousand items of widely varying costs and demand rates. The items may also differ considerably in their importance.

The system is to be operated with a continuous review inventory policy, so the problem is to determine reorder points and order quantities for each of the items so as to minimize the (expected) total number of shortages per unit time. The more traditional objective of operating the system to minimize the sum of the holding, shortage, and order costs is not considered in this report. For a more complete discussion of reorder point inventory models see Chapter 4 of [2].

Summarizing the essential features of the problem discussed here:

1. The objective is to minimize the expected number of shortages per unit time.
2. There are two basic constraints; one on the average (expected) investment level, the other is a workload constraint and restricts the total (expected) number of orders which can be processed per unit time.
3. The control variables are the reorder points and order quantities for each item.
4. The leadtime demand density and the average demand rate are known.

Consider the problem P_1

$$\text{Min} \quad Z = \sum_{i=1}^{N} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x_i - r_i) f_i(x_i) dx$$

$$(P_1) \quad \text{subject to} \quad \sum_{i=1}^{N} c_i \ (r_i - u_i + Q_i/2) \leq I$$

$$\sum_{i=1}^{N} \lambda_1 / Q_i \leq R$$

$$r_i \ , Q_i \ \geq \ 0 \ , \ i = 1, \ldots, N.$$

The objective function is the expected number of shortages per unit time and the constraints are on the average investment level and on the expected number of orders per unit time.

In this report we treat each item as equally important to the inventory manager. Alternatively, one could define K_i $i=1,..,N$ to be the "essentiality" of item j and multiply the ith term of the objective function by K_i. The analysis that follows would be unchanged. All that would be required is to replace λ_i in P_i by a new quantity $\lambda_i' = \lambda_i K_i$.

Notation: N is the number of items,

λ_i is the average demand rate for item i,

$f_i(x_i)$ is the density of leadtime demand,

u_i is the mean leadtime demand,

Q_i is the reorder quantity,

r_i is the reorder point,

c_i is the cost of item i.

In this report we will deal with the basic problem P_1 and consider several ways in which it might be attacked. Section II considers this problem without the constraint on the expected number of orders and gives a dynamic programming formulation of the problem. Section III returns to the problem P_1 with the order constraint and shows how the optimal policies can be determined in this case. A modification of P_1 is discussed in Section IV. The modification consists of treating the order quantities Q_i as having been fixed (externally) and then determining the optimal reorder points r_i for each item. Two approaches are discussed for this problem and some computational results are given.

II. Investment Constraint Only.

We will deal first with P_1 without the order constraint. Thus we have:

$$\text{Min} \quad Z = \sum_{i=1}^{N} \lambda_i / Q_i \int_{r_i}^{\infty} (x_i - r_i) f_i(x_i) dx_i$$

$$(P_2) \quad \text{subject to} \quad \sum_{i=1}^{N} c_i \; (r_i - u_i + Q_i/2) \le I$$

$$r_i, \; Q_i \ge 0, \; i = 1, \ldots, N.$$

If a solution were available for each of the N single item problems giving $Z_i^\circ \; (I_i)$, the minimum expected number of shortages in item i given that its average investment level does not exceed I_i, then P_2 could be solved as an N - stage dynamic programming problem as we will show.

Thus we will investigate the possibility of finding $Z_i^\circ \; (I_i)$ for many different values of I_i.

Consider for some i

$$Z_i^\circ(I_i) = \min \ \lambda_i/Q_i \int_{r_i}^{\infty} (x_i - r_i) f_i(x_i) dx_i$$

$$(P_3) \qquad \text{subject to} \quad c_i(r_i - u_i + Q_i/2) \leq I_i$$

$$r_i \, , \, Q_i \geq 0 \, .$$

The application of dynamic programming to this problem will be discussed.

Rewrite P_3 as

$$Z_i^\circ(I_i) = \min \ p_2(d_2) \cdot p_1(d_1)$$

$$\text{subject to} \quad a_2 d_2 + a_1 d_1 \leq I_i + c_i u_i$$

$$d_2 \, , \, d_1 \geq 0 \, ,$$

where

$$d_2 = Q_i ,$$

$$p_2(d_2) = \lambda_i/d_2 ,$$

$$d_1 = r_i ,$$

$$p_1(d_1) = \int_{d_1}^{\infty} (x_i - d_1) f_i(x_i) dx_i ,$$

and

$$a_2 = c_i/2 , \quad a_1 = c_i .$$

The recursive equations for P_3 are:

$$f^1(y_1) = \min \; p_1(d_1)$$

subject to $0 \leq a_1 d_1 \leq y_1$,

$$f^2(y_2) = \min \left[p_2(d_2) \cdot f^1(y_i) \right]$$

subject to $0 \leq a_2 d_2 \leq y_2$,

$$y_1 = y_2 - a_2 d_2 .$$

The optimal value of d_1 will always be $d_1^\circ = y_i/a_1$ because $p_1(d_1)$ is a decreasing function of d_1. Thus

$$f^2(y_2) = \min \left[p_2(d_2) \cdot p_1\left(\frac{y_2 - a_2 d_2}{a_2}\right) \right]$$

$$\text{subject to} \quad 0 \le a_2 d_2 \le y_2 ,$$

and $f^2(y_2)$ can be tabulated for any desired values of y_2.
If we found $f^2(y_2)$ for $0 \le y_2 \le I_i + c_i u_i$, then we would have $Z_i^\circ (I_i)$ since $Z_i^\circ = f^2$.
Having found the function $Z_i^\circ (I_i)$ for each $i = 1,\ldots,N$, problem P_2 reduces to

$$Z^\circ(I) = \min Z = \sum_{i=1}^{N} Z_i^\circ (I_i)$$

$$\text{subject to} \quad \sum_{i=1}^{N} I_i \le I + \sum_{i=1}^{N} c_i u_i$$

$$I_i \ge 0$$

which can easily be solved for many values of I by dynamic programming using the following recursive equations:

$$f_1(x_1) = \max_{0 \le I_1 \le x_1} Z_1^\circ(I_1) ,$$

$$f_n(x_n) = \max_{0 \le I_n \le x_n} \{Z_n^\circ(I_n) + f_{n-1}(x_{n-1})\}, \quad n = 2,\ldots,N$$

where
$$x_{n-1} = x_n - I_n, \quad n = 1,\ldots,N$$

$$x_N = I + \sum c_i u_i \; .$$

The recursive solution to these equations yields $f_N(x_N)$ which is $Z^\circ(I)$. Some discussion of the computational feasibility of this approach follows. Note that once the functions $Z_i^\circ(I_i)$ are obtained, the problem is reduced to a one state variable dynamic programming problem with N stages. Since, in dynamic programming, the amount of computation rises linearly with the number of stages, a very large number of stages can be considered. The computations involved in determining $Z_i^\circ(I_i)$ for each i can be reduced somewhat because of practical considerations. In the process of determining the optimal value of d_2 for each value of y_2, it is probably not realistic to consider extremely large values of $d_2 = Q_i$ since practical considerations limit the size of the orders which would be placed. However, since the optimal value of d_1 or r_i is always y_1/a_1 we get $r_i = (y_2-a_2d_2)/a_1$ and unrealistically large values of r_i may be obtained for large values of y_2. This is simply a reflection of the fact that it is not realistic to consider allocating all (or large portions) of the investment level resource to a single item. We can use this fact to improve the computational efficiency in the problem solution.

Let r_i' be a lower limit on the reorder point for item i. If no other lower limit can easily be determined, zero can be used.

Q_i' be a lower limit on the order quantity for item i. For very expensive low-demand items, this limit might be zero, but for inexpensive high-demand items much larger limits can be set,

and let r_i'' and Q_i'' be the upper limits on these quantities.

Then since

$$d_1 = (y_2-a_2d_2)/a_1$$

or

$$y_2 = a_1d_1+a_2d_2$$

or

$$y_2 = c_i\ (r_i+Q_i/2)$$

The lowest value of y_2 which we need to consider is

$$y_2' = c_i \ (r_i'+Q_i'/2) \ ,$$

and the highest value is

$$y_2'' = c_i \ (r_i''+Q_i''/2) \ .$$

This reduction in the number of y_2 values which must be considered can greatly reduce the amount of computation involved in solving each of the N subproblems for $Z_i^{\circ}(I_i)$. The N stage dynamic programming problem can also be solved more efficiently using these upper and lower limits. For example, at stage n when tabulating the function $f_n(x_n)$ we need to consider only those combinations of x_n and I_n which give x_{n-1} satisfying

$$\sum_{i=1}^{n-1} c_i \ (r_i'-u+Q_i'/2) \ \leq \ x_{n-1} \ \leq \ \sum_{i=1}^{n-1} c_i \ (r_i''-u_i+Q_i''/2).$$

The reason for this is that if x_{n-1} were below the limit just given, then there would not be sufficient resources (investment level) available to meet the lower bound requirements for the remaining items. If the lower bounds r_i' , Q_i' are set unrealistically high so that it is not possible to meet them all with a given level of allowed average investment level I , this would be revealed by a violation of the inequality

$$\sum_{i=1}^{n-1} c_i(r_i'-u_i+Q_i'/2) \leq I \quad .$$

III. Investment and Order Constraints.

We return now to P_1 which includes the order constraint

$$\min \sum_{i=1}^{N} \lambda_i/Q_i \int_{r_i}^{\infty} (x_i - r_i) f_i(x_i) dx_i$$

$$(P_1) \text{ subject to } \sum_{i=1}^{N} c_i(r_i + Q_i/2 - u_i) \leq I$$

$$\sum_{i=1}^{N} \lambda_i/Q_i \leq R$$

$$r_i , Q_i \geq 0 .$$

(a) Two State Variable Dynamic Programming Formulations.

This problem can be solved directly as a two state variable dynamic programming problem, although for a large problem the computations could easily be prohibitive unless some special methods were employed to reduce computations. See for example [3]. We will first state the appropriate recursive equations for a two state variable dynamic programming approach and then discuss one method of reducing the computations. The recursive equations are:

$$f_1(x_1,y_1) = \min \left[\lambda_1/Q_i \int_{r_i}^{\infty} (x-r_i) f_i(x_i) dx \right]$$

subject to $\quad r_i, Q_i \geq 0$

$$c_1(r_i - u_i + Q_i/2) \leq x_1$$

and $\quad \lambda_1/Q_1 \leq y_1$,

$$f_n(x_n,y_n) = \min \left[\lambda_n/Q_n \int_{r_n}^{\infty} (x-r_n) f(x) dx + f_{n-1}(x_{n-1},y_{n-1}) \right]$$

subject to $\quad r_n, Q_n \geq 0$

$$c_n(r_n - u_n + Q_n/2) \leq x_n$$

$$\lambda_n/Q_n \leq y_n \qquad n = 2,\ldots,N,$$

and

(1)
$$x_{n-1} = x_n - c_n(r_n - u_n + Q_n/2)$$
$$y_{n-1} = y_n - \lambda_n/Q_n \quad , \quad n = 2,\ldots,N \ ,$$
$$X_N = I \ , \quad Y_N = R \ .$$

Thus by recursively tabulating the functions $f_n(x_n,y_n)$ along with the optimal decision $r_n^\circ(x_n,y_n)$ and $Q_n^\circ(x_n,y_n)$ for every possible combination of x_n, y_n we can eventually obtain $f_N(x_N,y_N) = f_N(I, R)$, the minimum expected number of shortages.

Tracing back through the optimal decisions using (1) we can determine r_i° , Q_i° , $i = 1,\ldots,N$. The difficulty with this approach is computational since $f_n(x_n,y_n)$ must be tabulated for all combinations of (x_n,y_n). This means that for each combination considered all feasible decisions r_n,Q_n must be examined to determine $r_n^\circ(x_n,y_n)$, $Q_n^\circ(x_n,y_n)$.

(b) A State Variable Reduction Method

One approach for reducing the computations is to introduce the second constraint of P_1 into the objective function [1]. This gives

$$\min \sum_{i=1}^{N} \lambda_i/Q_i \ \{ \int_{r_i}^{\infty} (x-r_i)f_i(x)dx - \Theta \}$$

$$(P_4) \quad \text{subject to} \quad \sum_{i=1}^{N} c_i(r_i-u_i+Q_i/2) \leq I$$

$$r_i \ , Q_i \geq 0 \ .$$

Problem P_4 can be handled exactly like P_3 . We have for each item

$$f^2(y_2,\Theta) = \min \{ p_2(d_2) \cdot \left[f^1\left(\frac{y_2-a_2d_2}{a_1}\right) - \Theta \right] \}$$

$$\text{subject to} \quad 0 \leq a_2d_2 \leq y_2 ,$$

which can be tabulated for any desired values of y_2 and a given Θ . The advantage of this method over the two-state variable approach in (a) is that, for a given Θ , the problem P_4 is reduced to a single state variable problem. When the solution to P_4 is obtained for a particular Θ , the constraint

$$\sum_{i=1}^{N} \lambda_i/Q_i \leq R$$

must be checked. Suppose

$$\sum_{i=1}^{N} \lambda_i/Q_i = k ,$$

then the present solution is optimal for the problem P_1 with R replaced by k . If k is not sufficiently close to R , another value of Θ must be selected and P_4 solved again. If Θ_1 yields $k_1 < R$ then a smaller value of Θ is needed since k is non-increasing in Θ , see [1] .

The above solution procedure will normally require trying several values of Θ , but useful information is obtained with each solution. An indication of the change in the optimal solution due to changes in R is obtained.

IV. Variable r Only.

In this section we discuss a variation to the original problem. Here, we consider the case where only the reorder points are subject to control while the reorder quantities Q_i are determined externally. The reorder quantities in this case may be set by any reasonable method. We discuss the case where each Q_i is selected to be proportional to the square root of its demand rate λ_i divided by its cost c_i. The constant of proportionality is set such that the constraint on the number of orders is binding. Thus, we let

$$Q_i(R) = \sqrt{\lambda_i/c_i}\; k$$

and select k such that

$$\sum_{i=1}^{N} \lambda_i/Q_i(R) = R .$$

This gives

$$\sum_{i=1}^{N} \frac{\lambda_i}{\sqrt{\lambda_i/c_i}}\, k = R \quad ,$$

or

$$k = \frac{\sum \sqrt{\lambda_i c_i}}{R} ,$$

so that

$$Q_i (R) = \sqrt{\lambda_i / c_i} \left[\sum_{i=1}^{N} \sqrt{\lambda_i c_i} / R \right] .$$

If we set Q_i in the way just described, the problem P_1 reduces to

$$\min \quad Z = \sum_{i=1}^{N} \lambda_i / Q_i (R) \int_{r_i}^{\infty} (x_i - r_i) f_i (x_i) dx_i$$

(P_5) subject to

$$\sum_{i=1}^{N} c_i r_i = I , \quad r_i \geq 0$$

where

(1) $$I' = I + \sum_{i=1}^{N} c_i (u_i - \frac{Q_i (R)}{2}) .$$

Later in this section we discuss two approaches to this problem.

An interesting feature of this approach is that if a problem is solved for some value of R and a wide range of I' values, then the solution can easily be obtained, or approximated, for other R values. Notice that in P_5 the constraint is given in terms of I'. The corresponding value of I is

$$(2) \qquad I = I' + \sum_{i=1}^{N} c_i \left(u_i - \frac{Q_i(R)}{2}\right)$$

Suppose P_5 has been solved for a range of I' values yielding the optimal reorder points $r_i^\circ(I')$, $i = 1,\ldots,N$, and the optimal objective function value $Z^\circ(I')$. This solution is also optimal for any pair of I and R values satisfying equation (2), for fixed I'. If we wish to consider fixed I and want the solution for $R = R_1$, equation (1) tells which value of I' gives the solution for that value of R. Similarly for fixed R, if we want the optimal solution for $I = I_1$, we simply look at the optimal solution to P_5 for

$$I_1' = I' + (I_1 - I)$$

where I' is the value previously used in P_5 and I is the corresponding actual investment level for the previous solution.

These relationships among the optimal solutions can easily be exploited in the dynamic programming approach discussed next.

(a) A Dynamic Programming Approach to the Variable r Only Problem.

The problem P_5 can be formulated as a dynamic programming problem. We let the stages correspond to the individual items and let the stage return function be

$$p_n(c_n r_n) = (\lambda_n / Q_n(R)) \int_{r_n}^{\infty} (x_n - r_n) f_n(x_n) dx_n \ .$$

The recursive equations are

$$F_n(x_n) = \max \{ \frac{\lambda_n}{Q_n(R)} \int_{r_n}^{\infty} (x_n - r_n) f_n(x_n) dx_n + F_{n-1}(x_{n-1}) \} \quad n = 1, \ldots, N$$

subject to $\qquad 0 \leq c_n r_n \leq x_n$

$$F_0(x_0) = 0 \ ,$$

and the stage transformations are

$$x_N = I'$$

$$x_{n-1} = x_n - c_n r_n \ , \ n = 1, \ldots, N.$$

A sample problem was solved using this approach. One goal was to assess the approximate computation times which could be expected using this approach. The problem was a ten item problem ($N = 10$) with the data shown in Table 1.

Item i	λ_i	μ_i	σ_i	c_i
1	5.12	2.82	3.86	2.40
2	416.64	249.98	163.60	0.06
3	11.52	5.76	5.52	2.24
4	170.72	98.16	94.30	0.06
5	4.00	1.20	1.37	0.01
6	15.36	8.83	10.96	0.33
7	416.00	312.00	385.50	0.32
8	1.00	0.55	0.68	1.30
9	2.08	0.94	0.87	4.60
10	3.20	1.44	2.08	0.07

$$X_i \sim N(u_i, \sigma_i^2)$$

Table 1 : Data for Sample Problem

Dynamic programming easily permits the solution to be obtained for any I' value less than or equal I'_{max} which, in our problem, was selected to be 606.5. This number was selected because for the R value used (15.00) the quantity

$$- \sum c_i \left(u_i - \frac{Q_i(R)}{2}\right)$$

appearing in equation (1) has a value of 106.5 and the solution to the sample problem was sought for I values ranging from 0 to 500 in steps of 100 .

Before the computation time for this problem is discussed we should discuss briefly the operation of the program.

At each stage $(1,\ldots,N)$ of the computation the state variable x_n ranges from XLOW to XHIGH in steps of DELX and for each value of the state variable. The decision variable r_n is stepped from zero to x_n is steps of DELD and for each value of r_n, the return function is computed (using the recursive equation) and compared to the current best value.

In this sample problem we used XLOW=0.0 , XHIGH=606.5, DELX=5.0, DELD=12.0.

Solution time on the IBM 360/67 was about 42 seconds for this problem. It should also be remembered that for each evaluation of the return function it was necessary to evaluate the quantity

$$\int_{r_n}^{\infty} (x-r_n) f_n(x) dx .$$

This was done using the relationships

$$\int_{r}^{\infty} (x-r) f(x) dx = \frac{\sigma}{\sqrt{2\pi}} e^{-((r-u)/\sigma)^2/2} + \frac{1}{\sqrt{2\pi}} (u-r) \int_{\frac{r-u}{\sigma}}^{\infty} e^{-y^2/2} dy$$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-((r-u)/\sigma)^2/2} + (u-r) \left[1/2 - 1/2 \operatorname{erf} \left(\frac{r-u}{\sigma\sqrt{2}}\right) \right]$$

where erf is the "error function" defined by

$$\operatorname{erf} (t/\sqrt{2}) = 2 \int_0^t \phi(y) dy \quad \text{and} \quad \phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} .$$

The computation was originally done for $R = 15$ and $I'_{max} = 606.5$ ($I\ max = 500.0$). The dynamic programming solution to this problem involves tabulation of $f_n(x_n)$ for values of $x_n = 0, 5.0, 10.0\ldots,610.0$ and for each $n = 1,\ldots,9$. At stage 10 it would be sufficient to determine $f_N(x_n)$ only for $x_n = I'_{max}$ if that were the only value of I' of interest. However, the solution for any value of $I' \leq I'_{max}$ can easily be obtained by tabulating the function $f_N(x_N)$ for x_N between 0 and I'_{max}. This was done (for $x_N = 0,5.0,\ldots,610.0$) and the solution is shown in Figure 1 for $R = 15$ and $I' = 206.5, 306.5,\ldots,606.5$ corresponding to $I' = 100.0, 200.0,\ldots,500.0$. For $R = 50$ we get

$$- \sum c_i \left(u_i - \frac{Q_i(R)}{2}\right) = 135.8$$

so that the solution for $R = 50$, $I = 500 - (135.8-106.5) = 470.7$ is the same as for $R = 15$, $I = 500$.

The optimal values of the decision variable are given in Figure 2.

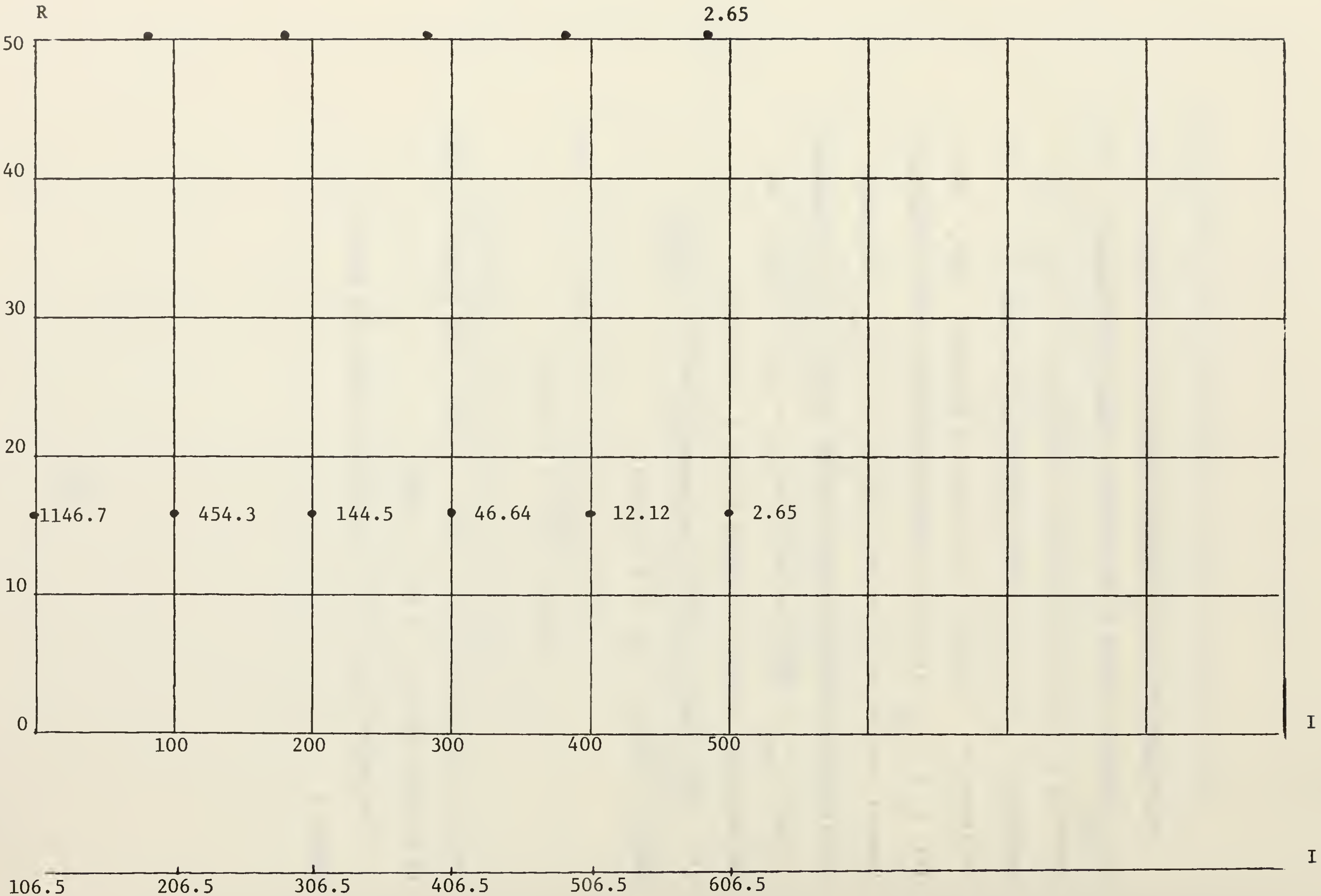

Figure 1. Results for the Sample Problem.
(Objective Function Values are Shown)

I	0	100	200	300	400	500
d_1°	0.00	0.00	0.00	0.00	7.60	10.00
d_2°	4.00	400.00	600.00	600.00	640.00	800.00
d_3°	0.00	0.00	0.00	5.36	10.71	16.07
d_4°	120.00	200.00	200.00	400.00	400.00	400.00
d_5°	0.00	0.00	0.00	0.00	0.00	0.00
d_6°	0.00	0.00	0.00	36.36	36.36	36.36
d_7°	225.00	525.00	787.50	975.00	1200.00	405.11
d_8°	0.00	0.00	0.00	0.00	0.00	0.00
d_9°	0.00	0.00	0.00	0.00	0.00	2.61
d_{10}°	0.00	0.00	0.00	0.00	0.00	0.00

Figure 2: Optimal Decisions For the Sample Problem, R = 15.

(b) Method of Convex Combinations.

For the problem with variable r only, an approach which deserves further investigation is the method of convex combinations which can be used to minimize a differentiable function $Z(\bar{r})$ subject to linear constraints on the variables. The method begins with a feasible point $\bar{r}_1$ and approximates the objective function by a hyperplane tangent at $\bar{r}_1$. The feasible point $\bar{y}_1$ providing the lowest value on the hyperplane is then determined (using linear programming). The actual objective function is then searched along the vector $\bar{y}_1 - \bar{r}_1$ to find the point $\bar{r}_2$ giving the lowest value of the objective function. From differentiability it follows that $Z(\bar{r})$ decreases locally at $\bar{r}_1$ in the direction $(\bar{y}_1-\bar{r}_1)$ unless $\bar{r}_1$ is the optimal solution. Thus

$$Z(\bar{r}_2) \leq Z(r_1)$$

and the process is repeated.

Using this method, the problem P_5

$$\min Z(\bar{r}) = \sum_{i=1}^{N} \frac{\lambda_i}{Q_i(R)} \int_{r_i}^{\infty} (x_i - r_i) f_i(x_i) dx_i$$

$$\text{subject to } \sum_{i=1}^{N} c_i r_i = I' \text{ , } r_i \geq 0 \text{ ,}$$

is replaced by

$$\min \sum_{i=1}^{N} r_i \frac{\partial Z(r_1)}{\partial r_i}$$

$$\text{subject to } \sum_{i=1}^{N} c_i r_i = I'$$

$$r_i \geq 0 \text{ .}$$

This problem is solved to yield a point $\bar{r}^* = \bar{y}_1$.

Notice that after the quantities $\partial Z_i(r_i)/\partial r_i$ are computed, the solution to this problem is trivial since there is a single linear constraint on the variables. We must find that item i for which the ratio $(\partial Z(r_1)/\partial r_i)/c_i$ is a minimum.

Then the function $Z(\bar{r}_1+\Theta(\bar{y}_1-\bar{r}_1))$ is searched for that Θ, $0 \leq \Theta \leq 1$ which minimizes Z. The resulting point $\bar{r}_1 + \Theta(\bar{y}_1-\bar{r}_1)$ becomes $\bar{r}_2$.

A program was written to solve the 10 item example given in the previous section. However, more investigation is needed before conclusive results can be obtained since a good deal of latitude is available in applying the method. Choices must be made about step sizes for searching along the vector $\bar{y}_1-\bar{r}_1$, and stopping rules must be selected to determine when to stop searching and repeat the process.

V. Conclusions.

Several approaches have been presented for the problem of determining continuous review inventory policies to minimize the expected total number of shortages in a multi-item system with known demand rates and leadtime demand densities. The general problem requires that the policies satisfy constraints on the average investment level and on the expected number of orders processed per unit time. Following is an assessment of the practicality of each of the approaches suggested.

The model of section two does not consider the order constraint and is, therefore, not really the problem of interest. If it is anticipated that the order constraint will, in fact, be a binding constraint then this approach will be of no use. However, since the shortage function

$$s(r_i Q_i) = \lambda_i / Q_i \int_{r_i}^{\infty} (x - r_i) f_i(x) \, dx$$

giving the expected number of shortages in item i does go through a minimum value for finite Q, the solution obtained from the solution to P_2 may satisfy the order constraint.

Whether this occurs can not be determined without actually solving the problem. The primary purpose of the discussion in Section II is that it provides the background for the approach in Section III where the order constraint is incorporated into the objective function using a multiplier.

Two methods are proposed in Section III for the general problem P_1. These are

(1) two state variable dynamic programming

(2) state variable reduction method

The first of these is computationally impractical for problems as large as the problem of real interest with 100,000 items. On the other hand, if the items can be somehow grouped according to demand rates and costs so the same r and Q is used for each item in the group, then the two state variable dynamic programming could be used to determine how to divide the resources (I and R) among the groups.

The state variable reduction method partially overcomes the computational difficulty of (a) but for extremely large problems the computations would still be lengthy since the problem normally has to be solved for several values of the multiplier before an appropriate value is found. Even so this approach would be preferred to (a), and it could also be used to even greater advantage if the items are grouped as discussed above.

Because of the linearity of the constraints, the method of convex combinations also appears promising and should be further investigated. It is known that the method provides a sequence of points which does converge to a constraint stationary point. There is no particular difficulty in investigating the method except that computational experience is needed to determine the rate of convergence. This rate will depend upon the starting point and the search method employed after the direction to move has been determined. An important feature of this method is that it moves to successively better solutions, and if computation time is limited, the computation will at least terminate with a feasible solution better than any previously available. As an immediate practical application of this, one could start with any solution currently available (for example the existing policy) and move to a better feasible solution.

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13. ABSTRACT

A formulation is given for minimizing the expected total number of shortages in a multi-item inventory system under a continuous review policy. The variable reorder levels and order quantities must be set to meet constraints on the average expected investment in on hand inventory and on the expected number of orders placed per unit time. Several approaches to the problem are presented.

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